# The Order Bound on the Minimum Distance of the One-Point Codes Associated to a Garcia-Stichtenoth Tower of Function Fields

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**Abstract**

Garcia and Stichtenoth discovered two towers of function fields that meet the Drinfeld-Vlăduţ bound on the ratio of the number of points to the genus. For one of these towers, Garcia, Pellikaan and Torres derived a recursive description of the Weierstrass semigroups associated to a tower of points on the associated curves. In this article, a non-recursive description of the semigroups is given and from this the enumeration of each of the semigroups is derived as well as its inverse. This enables us to find an explicit formula for the order (Feng-Rao) bound on the minimum distance of the associated one-point codes.

**Keywords:** Numerical semigroup, Garcia-Stichtenoth tower.

## Introduction

Let $\mathbb{N}_0$ denote the set of all non-negative integers. A *numerical semigroup* is a subset $\Lambda$ of $\mathbb{N}_0$ containing 0, closed under summation and with finite complement in $\mathbb{N}_0$. The *enumeration* of $\Lambda$ is the unique increasing bijective map $\lambda : \mathbb{N}_0 \longrightarrow \Lambda$. Usually $\lambda_i$ is used instead of $\lambda(i)$. Given a numerical semigroup $\Lambda$ with enumeration $\lambda$ define the sequence $\nu_i$ by

$$\nu_i = |\{j \in \mathbb{N}_0 : \lambda_i - \lambda_j \in \Lambda\}|.$$

This sequence has become of great importance in the theory of one-point algebraic-geometry codes. For a sequence of one-point codes on a curve, where the one point is $P$, the numerical semigroup considered is that of the pole orders in $P$ of functions having only poles in $P$, usually named as the *Weierstrass semigroup* of

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$P$. The sequence $\nu_i$ is used to define the *order bound* on the minimum distance of one-point algebraic-geometry codes:

$$\delta_i = \min\{\nu_j : j > i\}.$$

The order bound, also known as Feng-Rao bound, is a lower bound on the minimum distance of the $i$-th one-point code on $P$. In this case the numerical semigroup is the Weierstrass semigroup associated to $P$. Details can be found in [1, 2, 3].

Garcia and Stichtenoth first gave in [4, 5] two towers of function fields attaining the Drinfeld-Vlăduţ bound, which became of great importance in the area of algebraic coding theory. The tower in [5] is defined over the finite field with $q^2$ elements $\mathbb{F}_{q^2}$ for $q$ a prime power. It is given by

- $\mathcal{F}^1 = \mathbb{F}_{q^2}(x_1)$
- $\mathcal{F}^m = \mathcal{F}^{m-1}(x_m)$ with $x_m$ satisfying

$$x_m^q + x_m = \frac{x_{m-1}^q}{x_{m-1}^{q-1} + 1}.$$

It is shown in [5] that the number of its rational points is $N_q(\mathcal{F}^m) \geqslant (q^2 - q)q^{m-1}$ and that the genus $g_m$ of $\mathcal{F}^m$ is $g_m = (q^{\lfloor \frac{m+1}{2} \rfloor} - 1)(q^{\lceil \frac{m-1}{2} \rceil} - 1)$. Hence, the ratio between the genus $g(\mathcal{F}^m)$ and $N_{q^2}(\mathcal{F}^m)$ converges to $1/(q-1)$, the Drinfeld-Vlăduţ bound, as $m$ increases. From these curves one can construct asymptotically good sequences of codes.

For every function field $\mathcal{F}^m$ in the tower we distinguish the rational point $Q^m$ that is the unique pole of $x_1$. The Weierstrass semigroup $\Lambda^m$ at $Q^m$ in $\mathcal{F}^m$ was recursively described in [6]. Indeed, the semigroups are given recursively by

$$\begin{aligned} \Lambda^1 &= \mathbb{N}_0 \\ \Lambda^m &= q \cdot \Lambda^{m-1} \cup \{i \in \mathbb{N}_0 : i \geqslant q^m - q^{\lfloor \frac{m+1}{2} \rfloor}\}. \end{aligned} \tag{1}$$

Although these Weierstrass semigroups have been known for a long time, no explicit description of the order bound on the minimum distance for the associated one-point codes has appeared in the literature. Chen found certain bounds on the order bound for a given range and this enabled him to prove that some codes had distance larger than the order bound [7]. The main goal of this paper is to find the explicit description of the order bound from a deep analysis of these semigroups.

In Section 1 we give a non-recursive description of these semigroups. This description leads to explicit formulation of their enumerations as well as for the inverses of their enumerations. This is presented in Section 2. In Section 3 we find the explicit formula for the order bound on the minimum distance of the one-point codes associated to the tower of function fields.

# 1 Non-recursive description of the semigroups

In this section we will give a non-recursive description of the semigroups (1). The *conductor* of a numerical semigroup $\Lambda$ is the unique integer $c$ such that $c-1 \notin \Lambda$ and $c+\mathbb{N}_0 \subseteq \Lambda$. The *gaps* of $\Lambda$ are the elements in $\mathbb{N}_0 \setminus \Lambda$ and the *non-gaps* are the elements in $\Lambda$.

From now on we will use $c_m$ for the conductor of $\Lambda^m$, which is $q^m - q^{\lfloor \frac{m+1}{2} \rfloor}$, and $A_i = \{c_{2i-1}+j : j = 0, \dots, q^{i-1}(q-1)-1\} = \{j \in \mathbb{N}_0 : q^{2i-1}-q^i \leqslant j \leqslant q^{2i-1}-q^{i-1}-1\}$. Notice that these are not recursive definitions.

**Theorem 1**

$$\Lambda^m = \bigsqcup_{i=1}^{\lfloor \frac{m}{2} \rfloor} q^{m-2i+1} A_i \sqcup \{j \in \mathbb{N}_0 : j \geqslant c_m\}.$$

*Proof:* We will proceed by induction. The case $m = 1$ is obvious. Suppose $m > 1$. If $m$ is even, say $m = 2n$, then

$$\Lambda^{2n} = q\Lambda^{2n-1} \cup \{i \in \mathbb{N}_0 : i \geqslant c_{2n}\}.$$

By the induction hypothesis, since $c_{2n-1} = q^{2n-1} - q^n$ and $c_{2n} = q^{2n} - q^n$,

$$\begin{aligned}
\Lambda^{2n} &= \cup_{i=1}^{n-1} q^{2n-2i+1} A_i \cup q\{j \in \mathbb{N}_0 : j \geqslant q^{2n-1} - q^n\} \\
&\cup \{j \in \mathbb{N}_0 : j \geqslant q^{2n} - q^n\} \\
&= \cup_{i=1}^{n-1} q^{2n-2i+1} A_i \\
&\cup q\{j \in \mathbb{N}_0 : q^{2n-1} - q^n \leqslant j \leqslant q^{2n-1} - q^{n-1} - 1\} \\
&\cup \{j \in \mathbb{N}_0 : j \geqslant q^{2n} - q^n\} \\
&= \cup_{i=1}^{n} q^{2n-2i+1} A_i \cup \{j \in \mathbb{N}_0 : j \geqslant q^{2n} - q^n\} \\
&= \cup_{i=1}^{\lfloor \frac{(2n)}{2} \rfloor} q^{(2n)-2i+1} A_i \cup \{j \in \mathbb{N}_0 : j \geqslant c_{2n}\}.
\end{aligned}$$

If $m$ is odd, say $m = 2n+1$, then

$$\Lambda^{2n+1} = q\Lambda^{2n} \cup \{i \in \mathbb{N}_0 : i \geqslant c_{2n+1}\}.$$

By the induction hypothesis, since $c_{2n} = q^{2n} - q^n$ and $c_{2n+1} = q^{2n+1} - q^{n+1}$,

$$\begin{aligned}
\Lambda^{2n+1} &= \cup_{i=1}^{n} q^{2n-2i+2} A_i \cup q\{j \in \mathbb{N}_0 : j \geqslant q^{2n} - q^n\} \\
&\cup \{j \in \mathbb{N}_0 : j \geqslant q^{2n+1} - q^{n+1}\} \\
&= \cup_{i=1}^{n} q^{2n-2i+2} A_i \cup \{j \in \mathbb{N}_0 : j \geqslant q^{2n+1} - q^{n+1}\} \\
&= \cup_{i=1}^{\lfloor \frac{(2n+1)}{2} \rfloor} q^{(2n+1)-2i+1} A_i \cup \{j \in \mathbb{N}_0 : j \geqslant c_{2n+1}\}.
\end{aligned}$$

It remains to see that the unions are disjoint. Let us first see that the sets $q^{m-2i+1}A_i$ are pairwise disjoint. Indeed,

$$\begin{aligned}\max\left(q^{m-2i+1}A_i\right) &= q^{m-2i+1}(q^{2i-1}-q^{i-1}-1)\\ &= q^m - q^{m-i} - q^{m-2i+1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\min\left(q^{m-2(i+1)+1}A_{i+1}\right) &= q^{m-2i-1}(q^{2i+1}-q^{i+1})\\ &= q^m - q^{m-i}.\end{aligned}$$

This proves that the sets $q^{m-(2i-1)}A_i$ are pairwise disjoint.

Now, the maximum attained by these sets is $q^m - q^{m-\lfloor\frac{m}{2}\rfloor} - q^{m-2\lfloor\frac{m}{2}\rfloor+1}$. If $m$ is even then this equals $q^m - q^{\frac{m}{2}} - q = c_m - q$ while if $m$ is odd then this equals $q^m - q^{\frac{m+1}{2}} - q^2 = c_m - q^2$. $\square$

# 2 Enumeration of the semigroups

In this section we use the notations and results in Theorem 1 to find an explicit formula for the enumeration of the semigroups in (1) as well as a formula for its inverse.

**Theorem 2** *Let $\lambda$ be the enumeration of $\Lambda^m$.*

1. *The conductor $c_m$ is the image by $\lambda$ of $q^{\lfloor\frac{m}{2}\rfloor}-1$. That is, $c_m = \lambda_{q^{\lfloor\frac{m}{2}\rfloor}-1}$.*
2. *If $t \geqslant q^{\lfloor\frac{m}{2}\rfloor}-1$, then $\lambda_t = c_m + t - q^{\lfloor\frac{m}{2}\rfloor}+1$.*
3. *If $0 \leqslant t < q^{\lfloor\frac{m}{2}\rfloor}-1$, then $\lambda_t = q^{m-2l-1}(c_{2l+1}+t+1-q^l)$, where $l = \lfloor\log_q(t+1)\rfloor$.*

*Proof:*

1. For any positive integer $i$ it holds $|A_i| = q^{i-1}(q-1)$ and hence

$$\begin{aligned}\left|\bigsqcup_{i=1}^{\lfloor\frac{m}{2}\rfloor} q^{m-2i+1}A_i\right| &= (1+q+q^2+\cdots+q^{\lfloor\frac{m}{2}\rfloor-1})(q-1)\\ &= q^{\lfloor\frac{m}{2}\rfloor}-1.\end{aligned}$$

   Since $c_m$ is the first non-gap which is not in $\bigsqcup_{i=1}^{\lfloor\frac{m}{2}\rfloor} q^{m-2i+1}A_i$, $c_m = \lambda_{q^{\lfloor\frac{m}{2}\rfloor}-1}$.

2. If $t \geqslant q^{\lfloor\frac{m}{2}\rfloor}-1$, then $\lambda_t$ is in the subset $\{j \in \mathbb{N}_0 : j \geqslant c_m\}$ and for all $\lambda_k$ in this subset we have, $\lambda_{k+l} = \lambda_k + l$ for all $l \geqslant 0$. Taking $k = q^{\lfloor\frac{m}{2}\rfloor}-1$ and $l = t - q^{\lfloor\frac{m}{2}\rfloor}+1$ we get $\lambda_t = \lambda_{q^{\lfloor\frac{m}{2}\rfloor}-1} + t - q^{\lfloor\frac{m}{2}\rfloor}+1 = c_m + t - q^{\lfloor\frac{m}{2}\rfloor}+1$.

3. If $t < q^{\lfloor \frac{m}{2} \rfloor} - 1$, then the lemma below shows that $t$ may be uniquely expressed as $t = q^{l-1} + j - 1$ where $l = \lfloor \log_q(t+1) \rfloor + 1 \leqslant \lfloor m/2 \rfloor$ and $0 \leqslant j \leqslant q^{l-1}(q-1) - 1$. Since $|\bigsqcup_{k=1}^{l-1} q^{m-2k+1} A_k| = q^{l-1} - 1$, we have $\lambda_{q^{l-1}-1}$ is the first non-gap which is not in $\bigsqcup_{k=1}^{l-1} q^{m-2k+1} A_k$, namely $q^{m-2l+1} c_{2l-1}$. The next non-gaps after $q^{m-2l+1} c_{2l-1}$ are $q^{m-2l+1}(c_{2l-1}+i)$ with $i = 1, \ldots, q^{i-1}(q-1) - 1$. Therefore, $\lambda_t = q^{m-2l+1}(c_{2l-1} + j) = q^{m-2l+1}(c_{2l-1} + t + 1 - q^l)$.

□

**Lemma 3** *Any nonnegative integer $t$ may be uniquely expressed in the form*

$$t = q^{l-1} + j - 1$$

*for $1 \leqslant l$ and $0 \leqslant j \leqslant q^{l-1}(q-1) - 1$. Furthermore, $l = \lfloor \log_q(t+1) \rfloor$ and $j = t - q^{l-1} + 1$.*

*Proof:* Observe that

$$\begin{aligned} l = \lfloor \log_q(t+1) \rfloor + 1 &\iff q^{l-1} \leqslant t + 1 < q^l \\ &\iff 0 \leqslant t - q^{l-1} + 1 < q^{l-1}(q-1) \end{aligned}$$

Setting $j = t - q^{l-1} + 1$ shows that $t$ may be expressed as claimed, and that the choice of $l$ and $j$ are unique. □

We want to find now a formula for the inverse of the enumeration of a numerical semigroup. Given an integer $k$ and a numerical semigroup $\Lambda$ we define the *semigroup floor* of $k$ with respect to $\Lambda$ as the largest element in $\Lambda$ which is not larger than $k$. It is denoted $\lfloor k \rfloor_\Lambda$. In the next theorem we describe a formula not only to find the inverse of the enumeration of a numerical semigroup, but also to find the index for the semigroup floor of any integer.

**Theorem 4** *Let $\lambda$ be the enumeration of $\Lambda^m$ and let $k \geqslant 0$ be an integer. Let $c_i$ and $g_i$ be respectively the conductor and the genus of $\Lambda^i$. Then,*

$$\lambda^{-1}(\lfloor k \rfloor_{\Lambda^m}) = \begin{cases} k - g_m & \text{if } k \geqslant c_m, \\ q^{l-1} - 1 + \lfloor \frac{k}{q^{m-2l+1}} \rfloor - c_{2l-1} & \text{if } k < c_m, \end{cases}$$

*where $l = m + 1 - \lceil \log_q(q^m - k) \rceil$.*

*Proof:* The result for the case when $k \geqslant c_m$ is obvious. Suppose $k < c_m$. By Theorem 1, $\lfloor k \rfloor_{\Lambda^m}$ can be expressed as $q^{m-2l+1}(c_{2l-1}+i)$ with $i = 1, \ldots, q^{i-1}(q-1) - 1$ and with $l$ being the largest integer $l$ with $q^{m-2l+1} c_{2l-1} \leqslant \lfloor k \rfloor_{\Lambda^m}$. Notice that it is also the largest integer $l$ with $q^{m-2l+1} c_{2l-1} \leqslant k$. By substituting $c_{2l-1}$ by $q^{2l-1} - q^l$ one gets $l = m + 1 - \lceil \log_q(q^m - k) \rceil$. By Theorem 2. 3), it follows that $\lambda^{-1}(\lfloor k \rfloor_{\Lambda^m}) = q^{l-1} - 1 + \frac{\lfloor k \rfloor_{\Lambda^m}}{q^{m-2l+1}} - c_{2l-1}$. Now, $\frac{\lfloor k \rfloor_{\Lambda^m}}{q^{m-2l+1}} = \lfloor \frac{k}{q^{m-2l+1}} \rfloor$ is a consequence of the fact that $\lfloor k \rfloor_{\Lambda^m}$ belongs to $q^{m-2l+1} A_l$. □

# 3 The Order Bound on the Minimum Distance

From now on, $\lambda^m$ denotes the enumeration of $\Lambda^m$, $c_m, g_m$ are the conductor and the genus of $\Lambda^m$, $\nu_i^m$ is the $i$th value of the $\nu$-sequence corresponding to the semigroup $\Lambda^m$ and $\delta_i^m$ is the order bound on the minimum distance of the $i$th one point code associated to $\mathcal{F}^m$.

**Lemma 5** *$\nu_i^1 = i+1$ for all $i$. If $i > 1$ then*

$$\nu_i^m = \begin{cases} \nu_i^{m-1} & \text{if } i \leqslant c_m - g_m, \\ \nu^{m-1}_{\frac{i+g_m}{q} - g_{m-1}} & \text{if } c_m - g_m < i \leqslant 2c_m - g_m \text{ and } q \mid i + g_m, \\ 2 + 2(\lambda^m)^{-1}(\lfloor i + g_m - c_m - 1 \rfloor_{\Lambda^m}) & \text{if } c_m - g_m < i \leqslant 2c_m - g_m \text{ and } q \nmid i + g_m, \\ i - g_m + 1 & \text{otherwise.} \end{cases}$$

*Proof:* Notice that $c_m \geqslant qc_{m-1}$ implies that an element in $\Lambda^m$ is a multiple of $q$ if and only if it is in $q\Lambda^{m-1}$.

The case $i \leqslant c_m - g_m$ is equivalent to $\lambda_i^m \leqslant c_m$. From the inductive definition of $\Lambda^m$ it is clear that $\lambda_i^m = q\lambda_i^{m-1}$ and consequently, $\nu_i^m = \nu_i^{m-1}$.

The case $i \geqslant 2c_m - g_m$ is equivalent to $\lambda_i^m \geqslant 2c_m$. It is well known that for any semigroup $\Lambda$ with conductor $c$ and genus $g$ the sequence $\nu$ satisfies that $\nu_i = i - g + 1$ for all $i \geqslant 2c - g$.

We are left with the case $c_m \leqslant \lambda_i^m < 2c_m$. Suppose $\lambda_i^m = \lambda_j^m + \lambda_k^m$ with $\lambda_j^m \leqslant \lambda_k^m$. Then $\lambda_j^m < c_m$ so $\lambda_j^m \in q\Lambda^{m-1}$. If $q \mid \lambda_i^m$ then also $q \mid \lambda_k^m$, so $\lambda_k^m \in q\Lambda^{m-1}$. This shows that we have a one-to-one correspondence between $\{j \in \mathbb{N}_0 : \lambda_i^m - \lambda_j^m \in \Lambda^m\}$ and $\{j \in \mathbb{N}_0 : \lambda_{i'}^{m-1} - \lambda_j^{m-1} \in \Lambda^{m-1}\}$, where $q\lambda_{i'}^{m-1} = \lambda_i^m$, that is, $i' = (i+g_m)/q - g_{m-1}$. Thus $\nu_i^m = \nu_{i'}^{m-1}$.

If $q \nmid \lambda_i^m$ then also $q \nmid \lambda_k^m$. Consequently, $\lambda_k^m > c_m$ and $\lambda_j^m < \lambda_i^m - c_m$. One can see that each $\lambda_j^m \in \Lambda^m$ with $\lambda_j^m < \lambda_i^m - c_m$ yields a pair of elements in $\{j \in \mathbb{N}_0 : \lambda_i^m - \lambda_j^m \in \Lambda^m\}$. Thus $\nu_i = 2|\{\alpha \in \Lambda^m : \alpha < \lambda_i^m - c_m\}| = 2 + 2(\lambda^m)^{-1}(\lfloor i + g_m - c_m - 1 \rfloor_{\Lambda^m})$. $\square$

**Lemma 6** *The order bound on the minimum distance satisfies*

$$\delta_i^m = \begin{cases} 2 & \text{if } i \leqslant c_m - g_m, \\ \nu_{i+2}^m & \text{if } c_m - g_m < i \leqslant 2c_m - g_m - 2 \text{ and } q \mid i + 1 + g_m, \\ \nu_{i+1}^m & \text{otherwise.} \end{cases}$$

*Proof:* Notice that by Lemma 5, $\nu^m_{c_m-g_m+1}=2$ for all $m$. Thus, if $i\leqslant c_m-g_m$ then $\delta^m_i=2$.

By Lemma 5, $\nu^m$ is increasing in the subset $\{i\in\mathbb{N}_0 : c_m-g_m<i\leqslant 2c_m-g_m, q\nmid i+g_m\}\cup\{i\in\mathbb{N}_0 : i>2c_m-g_m\}$. Now it is enough to check that for $i$ such that $c_m-g_m<i\leqslant 2c_m-g_m-2$ and $q\mid i+1+g_m$, $\nu^m_{i+1}\geqslant\nu^m_{i+2}$. To see this we will show that $\{(j,k)\in\mathbb{N}_0\times\mathbb{N}_0, j\leqslant k : \lambda^m_j+\lambda^m_k=\lambda^m_{i+2}\}\subseteq\{(j,k)\in\mathbb{N}_0\times\mathbb{N}_0, j\leqslant k : \lambda^m_j+\lambda^m_{k-1}=\lambda^m_{i+1}\}$. Indeed, since $q\mid i+1+g_m$, $q\nmid i+2+g_m=\lambda^m_{i+2}$. So, $\lambda^m_{i+2}=\lambda^m_j+\lambda^m_k$ with $\lambda^m_j\leqslant\lambda^m_k$ is only possible if $\lambda^m_j<c_m$ and $\lambda^m_k>c_m$. In this case, $\lambda^m_{i+1}=\lambda^m_{i+2}-1=\lambda^m_j+\lambda^m_k-1=\lambda^m_j+\lambda^m_{k-1}$. $\square$

**Theorem 7** *The order bound on the minimum distance is*

$$\delta^m_i=\begin{cases} 2 & \text{if } i\leqslant c_m-g_m,\\[2ex] 2q^{m-\alpha}+2\frac{i+1+g_m-c_m}{q^{2\alpha-m-1}}-2c_{2m-2\alpha+1}, & \\ \text{where } \alpha=\lceil\log_q(q^m-i-1+c_m-g_m)\rceil & \\ \qquad\qquad \text{if } c_m-g_m<i\leqslant 2c_m-g_m-2, & \\[2ex] i-g_m+2 & \text{if } i>2c_m-g_m-2.\end{cases}$$

*Proof:* The cases $i\leqslant c_m-g_m$ and $i>2c_m-g_m-2$ follow directly from Lemma 6 and Lemma 5.

Suppose $c_m-g_m<i\leqslant 2c_m-g_m-2$. We will show that $\delta^m_i=2q^{l-1}+2\frac{i+1+g_m-c_m}{q^{m-2l+1}}-2c_{2l-1}$, where $l=m+1-\lceil\log_q(q^m-i-1+c_m-g_m)\rceil$. Substituting $l=m+1-\alpha$ with $\alpha$ as defined above gives the desired result.

If $q\mid i+1+g_m$, then

$$\begin{aligned}\delta^m_i&=\nu^m_{i+2}\\&=2+2(\lambda^m)^{-1}(\lfloor i+1+g_m-c_m\rfloor_{\Lambda^m})\\&=2q^{l-1}+2\frac{i+1+g_m-c_m}{q^{m-2l+1}}-2c_{2l-1}.\end{aligned}$$

where $l=m+1-\lceil\log_q(q^m-i-1+c_m-g_m)\rceil$. Here we have used Lemma 6, Lemma 5, and Theorem 4.

For $q\nmid i+1+g_m$,

$$\delta^m_i=\nu^m_{i+1}=2+2(\lambda^m)^{-1}(\lfloor i+g_m-c_m\rfloor_{\Lambda^m}).$$

However, $\lfloor i+1+g_m-c_m\rfloor_{\Lambda^m}=\lfloor i+g_m-c_m\rfloor_{\Lambda^m}$ since in this case $i+1+g_m-c_m<c_m$ and $q\nmid i+1+g_m-c_m$. Thus, we can use the same formula as above for $\delta^m_i$. $\square$

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